J. Engg. and Appl. Sci. Vol.36 No,2 July - December 2017 ISSN 1023-862X - eISSN 2518-4571

SOME AGGREGATION OPERATORS FOR BIPOLAR-VALUED HESITANT FUZZY INFORMATION BASED ON EINSTEIN OPERATIONAL LAWS

Tahir Mahmood[1], Kifayat Ullah[2], Murad Ullah[3*], Naeem Jan[4], Irfan Deli[5], Qaisar Khan[6]

ABSTRACT

This article is based on Einstein operations for bipolar-valued hesitant fuzzy sets (BVHFSs). We extend the concept of Einstein operators for BVHFSs by defining bipolar-valued hesitant fuzzy Einstein weighted averaging (BPVHFEWA) operators and weighted geometric (BPVHFEWG) operators. Similarly, we define ordered weighted averaging operators and hybrid operators i.e. BVHFEOWA operators, BVHFEOWG operators, BVHFEHA operator and BVHFEHG operator. Further, these operators are applied in decision making DM) problems.

KEYWORDS: *Bipolar-valued hesitant fuzzy sets. Einstein operations. Aggregation operators. Decision making.*

INTRODUCTION

In the field of fuzzy mathematics decision making (DM) has a great importance. In fact, DM plays an important role in every walk of our daily life. At different stages of his life any person would have to take decision about his career. DM problem are so complex that ordinary mathematical techniques are not be able to deal with it.

(Zadeh,1965) gave rise to a new concept known as fuzzy set (FS). FSs have been discovered to deal with any kind of uncertainties and it plays a successful role in dealing with any kind of fuzziness. Soon after the discovery of FSs different new advance extensions of FSs are developed. Some famous extensions of FSs are hesitant fuzzy sets (HFSs) (Torra et al., 2009 &2010), intuitionistic fuzzy sets (IFSs), inter-valued fuzzy sets (IVFSs) (Dubois and Prade, 2010), bipolar-valued fuzzy sets (BVFSs) (Lee, 2000). Almost all of these extensions of FSs are somehow related to DM process. In fact in DM we need to look at the basic properties of concerned FS. For some related work one may refer to (Chauhan and Vaish, 2014; De et al., 2000; Farhadinia, 2013; Herrera-Viedma et al., 2007; Hong and Choi, 2000; Klutho, 2013; Li, 2005; Liu and Wang 2007; Abdullah et al., 2014).

Among different extensions of FSs, the theory of HFSs is of great interest. It was presented by Torra and Narukawa (2009) and it was indeed a unique idea of its kind that HFSs have affiliation degrees in the form of finite elements. This was such an interesting theory that most of the scientist took great interest in HFSs and soon a score function was defined in order to measure a HFS. (Chen, 2011; Chen et al., 2013; Chen and Xu, 2014) presented a function for the deviation degree HFSs. (Xia et al., (2013) used HFSs in DM process. For related work one may refer to (Liu et al., 2017; Neves and Livet 2008; Shabir and Israr 2008; Torra and Narukawa 2009; Torra, 2010; Xia and Xu, 2011; Xia et al., 2013; Xu, 2004; Xu and Yager, 2006; Xu, 2007; Xu and Jian, 2007; Ye, 2007; Ye, 2009).

Mahmood et al., (2016), presented a new theory known as BVHFS. BVHFS is a new enhanced extension of FSs developed by combining BVFSs and HFSs. Membership functions of BVHFSs defined in terms of finite values from [-1,1]. The positive membership function is a finite set whose values are taken from [0,1] conveying the level of satisfaction of an element. The negative membership function is a finite set whose values are from [-1,0] conveying the counter satisfaction level of an element. Mahmood et al., (2016), developed some basic theory for BVHFS and proved useful results of interest. Some aggregation operators were also defined for BVHFS and then applied in DM process in Mahmood et al., (2017). For more work (Mahmood et al., 2013; Mahmood et al., 2016a; Mahmood et al., 2017, Mahmood et al., 2016b).

The concept of Einstein operations was laid down by (Wang and Liu, 2011) at first. They defined Einstein operations for IFSs and they also developed theory of aggregation operators based on Einstein operations. They also defined Einstein T-norm and T-conorm and developed the theory of Einstein operations. Motivated

1,2,4,5,6 Department of Mathematics & Statistics, International Islamic University, Islamabad.
3 Department of Mathematics, Islamia College University, Peshawar, KPK, Pakistan.*

by the concepts of (Wang and Liu, 2011) soon, Dejian Yu take the concept of Einstein operations in the field of HFSs and define some very basic operation for HFSs based on Einstein operations. Yu (2014) also defined several aggregation operators of HFSs based on Einstein operations and then apply these concepts to a MADM problem.

We in this treatise, define Einstein operations for BVHFS. We get inspired by Yu (2014) defined some aggregation operators for BVHFS based on Einstein operation. We prove some interesting results and support our ideas with the help of few examples. Finally we apply the newly defined aggregation operators to MADM problem and find these operations very beneficial in DM process.

Starting with introduction, this article has five sections. In section two, we present some previous concepts related to our work which includes HFSs, IFSs, BVHFSs and some defined Einstein operations. In section three we define Einstein operation for BVHFSs and present an example in support of our newly defined operations. Section four deals with aggregation operators for BVHFSs. We also proved some useful results in this section along with few examples. The last section consists of DM process briefly and a MADM problem solved by using aggregation operators defined in section four. Finally, we conclude our article by adding some concluding remarks.

PRELIMINARIES

This section consists of some previous concepts to support our new work. It contains concepts related to BVHFSs and their operations along with Einstein operations for HFSs.

Definition 1: (Atanassov, 1986**)**

An IFS on a set X is of the form where $\mu_\alpha : X \to [0,1] \forall x \in X$ is known as a membership function and $\nu_\alpha : X \to [0,1] \forall x \in X$ is known as a non-membership function provided that $0 \le \mu_\alpha(x) + \nu_\alpha(x) \le 1 \forall x \in X$. Here μ_a (x) and v_a (x) represents the degree of membership and non-membership of element $x \in X$. For our ease $a=(\mu_a, v_a)$is called an intuitionistic fuzzy number (IFN).

Wang and Liu bring forth some new operations for IFVs based on Einstein operations which are described below:

Definition 2: (Wang and Liu 2011**)**

Let $\alpha, \alpha_1, \alpha_2$ be three IFVs then we define some Einstein operations on $\alpha, \alpha_1, \alpha_2$ as follows:

1. $\alpha_1 \oplus \alpha_2 = \left(\dfrac{\mu_{\alpha_1} + \nu_{\alpha_1}}{1 + \mu_{\alpha_1} . \nu_{\alpha_1}}, \dfrac{\mu_{\alpha_2} . \nu_{\alpha_2}}{1 + \left(1 - \mu_{\alpha_2}\right)\left(1 - \nu_{\alpha_2}\right)} \right)$
2. $\alpha_1 \otimes \alpha_2 = \left(\dfrac{\mu_{\alpha_1} . \nu_{\alpha_1}}{1 + \left(1 - \mu_{\alpha_1}\right)\left(1 - \nu_{\alpha_1}\right)}, \dfrac{\mu_{\alpha_2} + \nu_{\alpha_2}}{1 + \mu_{\alpha_2} . \nu_{\alpha_2}} \right)$
3. $\rho^{\cdot} \alpha = \left(\dfrac{\left(1 + \mu_\alpha\right)^{\rho^{\cdot}} - \left(1 - \mu_\alpha\right)^{\rho^{\cdot}}}{\left(1 + \mu_\alpha\right)^{\rho^{\cdot}} + \left(1 - \mu_\alpha\right)^{\rho^{\cdot}}}, \dfrac{2 \mu_\alpha{}^{\rho^{\cdot}}}{\left(2 - \mu_\alpha\right)^{\rho^{\cdot}} + \mu_\alpha{}^{\rho^{\cdot}}} \right) \rho^{\cdot}$ is any constant
4. $\alpha^{\rho^{\cdot}} = \left(\dfrac{2 \mu_\alpha{}^{\rho^{\cdot}}}{\left(2 - \mu_\alpha\right)^{\rho^{\cdot}} + \mu_\alpha{}^{\rho^{\cdot}}}, \dfrac{\left(1 + \mu_\alpha\right)^{\rho^{\cdot}} - \left(1 - \mu_\alpha\right)^{\rho^{\cdot}}}{\left(1 + \mu_\alpha\right)^{\rho^{\cdot}} + \left(1 - \mu_\alpha\right)^{\rho^{\cdot}}} \right) \rho^{\cdot}$ is any constant

Definition 3: (Torra and Narukawa, 2009; Torra, 2010**)**

Let X be any set. Then a HFS on X in the form of a function H that when applied to X gives us few values in [0,1]. A HFS Hon X is denoted as $H = \{x, h(x) \mid \forall x \in X\}$ where h(x) is a finite set whose values are form [0,1]. The set of all HFSs is expressed by H'. Here h(x) is known as HFE.

Xia and Xu developed some operations on HFSs which are given as follows:

1. $H^{\rho^{\cdot}}(x) = \left\{ m^{\rho^{\cdot}} : m \in H(x) \right\}$
2. $\rho^{\cdot} H(x) = \left\{ 1 - (1 - m)^{\rho^{\cdot}} : m \in H(x) \right\}$

Definition 4: (Yu, 2014**)**

According to Dejian Yu, Einstein operations for three HFEs are defined as follows:

1. $H_1 \oplus H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_1} \left\{ \dfrac{m_1 + m_2}{1 + m_1 m_2} \right\}$ is any constant
2. $H_1 \otimes H_2 = \bigcup_{m_1 \in H_1, m_2 \in H_1} \left\{ \dfrac{m_1 m_2}{1 + (1 - m_1)(1 - m_2)} \right\}$ is any constant
3. $\rho^{\cdot} H = \bigcup_{m \in H} \left\{ \dfrac{(1 + m)^{\rho^{\cdot}} - (1 - m)^{\rho^{\cdot}}}{(1 + m)^{\rho^{\cdot}} + (1 - m)^{\rho^{\cdot}}} \right\} \rho^{\cdot}$ is any constant
4. $H^{\rho^{\cdot}} = \bigcup_{m \in H} \left\{ \dfrac{2 m^{\rho^{\cdot}}}{(2 - m)^{\rho^{\cdot}} + m^{\rho^{\cdot}}} \right\} \rho^{\cdot}$ is any constant

Definition 5: (Mahmood et al., 2016a)

For any set X, the BVHFS $\mathfrak{B}$ on some domain of X is denoted and defined by:

$$\mathfrak{B} = \{x, (H^{+}(x), H^{-}(x) : x \in X\}$$

Where H^{+}:X $\rightarrow$[0,1] is a finite set whose values are from [0,1] conveying the satisfaction level of "x" to BVHFS $\mathfrak{B}$ and H^{-}:X$\rightarrow$[−1,0] is a finite set whose values are from [−1,0] conveying the dissatisfaction level of "x" to BVHFS $\mathfrak{B}$. Here $H = \{H^{+}(x), H^{-}(x)\}$ is a BVHFE. The set of all BVHFEs is denoted by Φ.

Consider two BVHFSs.

$$\tilde{\mathfrak{A}} = \{(x, (H^{+}_{\tilde{\mathfrak{A}}}(x), (H^{-}_{\tilde{\mathfrak{A}}}(x)) : x \in X\}$$

$$\mathfrak{B} = \{(x, (H^{+}_{\mathfrak{B}}(x), (H^{-}_{\mathfrak{B}}(x)) : x \in X\}$$

Some operations for BVHFSs are as follows:

$$(\tilde{\mathfrak{A}} \oplus \mathfrak{B})(x) = \{m_1 + m_2 - m_1 m_2 : m_1 \in H^{+}_{\tilde{\mathfrak{A}}}(x),\ m_2 \in H^{+}_{\mathfrak{B}}(x), \\ -(m_1 m_2) : m_1 \in H^{-}_{\tilde{\mathfrak{A}}}(x),\ m_2 \in H^{-}_{\mathfrak{B}}(x)\}$$

$$(\tilde{\mathfrak{A}} \otimes \mathfrak{B})(x) = \{m_1 m_2 : m_1 \in H^{+}_{\tilde{\mathfrak{A}}}(x),\ m_2 \in H^{+}_{\mathfrak{B}}(x), \\ -(-m_1 - m_2 - m_1 m_2) : m_1 \in H^{-}_{\tilde{\mathfrak{A}}}(x),\ m_2 \in H^{-}_{\mathfrak{B}}(x)\}$$

For any $\rho^{\cdot} > 0$

Definition 6: (Mahmood et al., 2016a)

Let $H = \langle H^{+}, H^{-} \rangle$ be a BVHFE. Then score function S of H is defined as:

$$\mathcal{S}(H) = \frac{1}{\ell_H}\left(\xi_H^{+} + \xi_H^{-}\right)$$

Where ξ_H^{+} is the total of positive membership grades and ξ_H^{-} is the total of negative membership grades and $\mathcal{S}(H) \in [-1,1]$.

Remark 1: (Mahmood et al., 2016a)

Length H^{+} and H^{-} of are not necessarily equal.

For two BVHFEs H_1 and H_2 if

$\mathcal{S}(H_1) < \mathcal{S}(H_2)$, then H_1 is smaller than H_2 i.e. $H_1 < H_2$.

$\mathcal{S}(H_1) > \mathcal{S}(H_2)$, then H_1 is bigger than H_2 i.e. $H_1 > H_2$.

$\mathcal{S}(H_1) = \mathcal{S}(H_2)$, then H_1 is indifferent (similar) to H_2 denoted by $H_1 \sim H_2$.

Yu, (2014), defined Einstein operation for HFSs. He also defines several aggregation operators for HFSs. Enhancing the same concept, we developed the same operation for BVHFSs. We are also successful in defining several aggregation operators for BVHFSs and then we apply these operators to a MADM problem and get quite useful result.

Einstein Operations for Bipolar-Valued Hesitant Fuzzy Sets

Definition 7:

Let H, H_1, H_2 three BVHFEs then Einstein operations are defined as:

1. $H_1 \oplus H_2 = \bigcup_{m_1 \in H_1,\ m_2 \in H_1} \left\{ \left\{ \frac{m_1 + m_2}{1 + m_1 m_2} \right\}, \left\{ \frac{-(-m_1)(-m_2)}{-1-(-1-m_1)(-1-m_2)} \right\} \right\}$

2. $H_1 \otimes H_2 = \bigcup_{m_1 \in H_1,\ m_2 \in H_1} \left\{ \left\{ \frac{m_1 m_2}{1+(1-m_1)(1-m_2)} \right\}, \left\{ \frac{-(-m_1 - m_2)}{-(-1-(-m_1)(-m_2))} \right\} \right\}$

3. $\rho^{\cdot} H = \bigcup_{m \in H} \left\{ \left\{ \frac{(1+m)^{\rho^{\cdot}} - (1-m)^{\rho^{\cdot}}}{(1+m)^{\rho^{\cdot}} + (1-m)^{\rho^{\cdot}}} \right\}, \left\{ \frac{-2(-m)^{\rho^{\cdot}}}{(-(-2-m))^{\rho^{\cdot}} - (-(-m)^{\dot{\rho}})} \right\} \right\}$ $\rho^{\cdot}$ is any constant

4. $H^{\rho^{\cdot}} \bigcup_{m \in H} \left\{ \left\{ \frac{2m^{\rho^{\cdot}}}{(2-m)^{\rho^{\cdot}} + m^{\rho^{\cdot}}} \right\}, \left\{ \frac{-((-(-1-(-m)))^{\rho^{\cdot}} - (-(-1-m))^{\rho^{\cdot}})}{(-(-2-m))^{\rho^{\cdot}} - (-(-m)^{\rho^{\cdot}})} \right\} \right\}$ $\rho^{\cdot}$ is any constant

Example 1:

Let $H = \{\{0.1, 0.2\}, \{-0.3, -0.2\}\}$, $H_1 = \{\{0.5, 0.6\}, \{-0.2, -0.1\}\}$

$H_2 = \{\{0.9, 0.8\}, \{-0.5, -0.4\}\}$ then

$$H_1 \oplus H_2 = \{\{0.965517, 0.928571, 0.974026, 0.945946\}, \{-0.07143, -0.05405, -0.03448, -0.02597\}\}$$

$$H_1 \otimes H_2 = \{\{0.428571, 0.363636, 0.519231, 0.444444\}, \{-0.63636, -0.55556, -0.57143, -0.48077\}\}$$

When $\dot{\rho} = 2$ then

$$\dot{\rho} H = \{\{0.19802, 0.384615\}, \{-0.06429, -0.025\}\}$$

$$H^{\rho^{\cdot}} = \{\{0.004535, 0.016529\}, \{-0.42857, -0.25\}\}$$

Aggregation Operators

Based on Einstein operational law, some aggregation operators for BPVHF are developed in this section. The proposed operators are briefly explored with the examples. In our further study, the following should be kept in mind:

$w_i^T = (w_1, w_2, \ldots w_n)^T$ denotes the weight vectors.

$w^{\cdot}=\left(w^{\cdot}_1,w^{\cdot}_2,\ldots \dot{w}_n\right)^T$ denotes aggregation-associated weight vector.

H_i represents some BVHFEs.

$\left(i,j,k=1,2,3,4\ldots n\right)$ be some set of indices.

Definition 8:

For some BVHFEs H_i and their weight vectors $w^{\cdot T}_i$ such that $w^{\cdot}_i\in[0,1]$ and $\sum_{j=1}^{n}w^{\cdot}_j=1$, than BVHFEWA operator is a function $\Psi^n\to\Psi$ such that $BPVHFEWA\left(H_1,H_2,\ldots H_n\right)=\oplus_{j=1}^{n}\left(w^{\cdot}_jH_j\right)$

if we choose the weight vector $w^{\cdot}=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\frac{1}{n},\ldots\frac{1}{n}\right)^T$ then BVHFEWA operator becomes BVHFEA operator i.e. $BVHFEA\left(H_1,H_2,\ldots H_n\right)=\oplus_{j=1}^{n}\left(\frac{1}{n}H_j\right)$

Using the above definitions, by applying induction technique on 'n' some results are proved.

Theorem 1:

For some BVHFEs H_i. Their aggregated value determined by using BVHFEWA operator is a BVHFE and

$$BVHFEWA\left(H_1,H_2,\ldots H_n\right)=\bigcup_{m_j\in H_j}\left\{\left\{\frac{\prod_{j=1}^{n}(1+m_j)^{w_j}-\prod_{j=1}^{n}(1-m_j)^{w_j}}{\prod_{j=1}^{n}(1+m_j)^{w_j}-\prod_{j=1}^{n}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{n}(-m_j)^{w_j}}{\prod_{j=1}^{n}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{n}(-m_j)^{w_j})}\right\}\right\}$$

Proof:

The first part of this theorem is obvious. To prove the second part, we need to use induction. First, the result is proved true for n=2.

When n=2

$$w^{\cdot}_1H_1=\left\{\left\{\frac{\left(1+m_1\right)^{w^{\cdot}_1}-\left(1-m_1\right)^{w^{\cdot}_1}}{\left(1+m_1\right)^{w^{\cdot}_1}+\left(1-m_1\right)^{w^{\cdot}_1}}\right\},\left\{\frac{-2\left(-m_1\right)^{w^{\cdot}_1}}{\left(-\left(-2-m_1\right)\right)^{w^{\cdot}_1}-\left(-\left(-m_1\right)^{w^{\cdot}_1}\right)}\right\}\right\}$$

$$w^{\cdot}_2H_2=\left\{\left\{\frac{\left(1+m_2\right)^{w^{\cdot}_2}-\left(1-m_2\right)^{w^{\cdot}_2}}{\left(1+m_2\right)^{w^{\cdot}_2}+\left(1-m_2\right)^{w^{\cdot}_2}}\right\},\left\{\frac{-2\left(-m_2\right)^{w^{\cdot}_2}}{\left(-\left(-2-m_2\right)\right)^{w^{\cdot}_2}-\left(-\left(-m_2\right)^{w^{\cdot}_2}\right)}\right\}\right\}$$

Now $w^{\cdot}_1H_1\oplus w^{\cdot}_2H_2$

$$=\left\{\begin{array}{l}\left\{\frac{\left(1+m_1\right)^{w^{\cdot}_1}-\left(1-m_1\right)^{w^{\cdot}_1}}{\left(1+m_1\right)^{w^{\cdot}_1}+\left(1-m_1\right)^{w^{\cdot}_1}}\right\},\\ \left\{\frac{-2\left(-m_1\right)^{w^{\cdot}_1}}{\left(-\left(-2-m_1\right)\right)^{w^{\cdot}_1}-\left(-\left(-m_1\right)^{w^{\cdot}_1}\right)}\right\}\end{array}\right\}\oplus\left\{\begin{array}{l}\left\{\frac{\left(1+m_2\right)^{w^{\cdot}_2}-\left(1-m_2\right)^{w^{\cdot}_2}}{\left(1+m_2\right)^{w^{\cdot}_2}+\left(1-m_2\right)^{w^{\cdot}_2}}\right\},\\ \left\{\frac{-2\left(-m_2\right)^{w^{\cdot}_2}}{\left(-\left(-2-m_2\right)\right)^{w^{\cdot}_2}-\left(-\left(-m_2\right)^{w^{\cdot}_2}\right)}\right\}\end{array}\right\}$$

$$=\left\{\begin{array}{l}\left\{\frac{\left\{\frac{\left(1+m_1\right)^{w^{\cdot}_1}-\left(1-m_1\right)^{w^{\cdot}_1}}{\left(1+m_1\right)^{w^{\cdot}_1}+\left(1-m_1\right)^{w^{\cdot}_1}}\right\}+\left\{\frac{\left(1+m_2\right)^{w^{\cdot}_2}-\left(1-m_2\right)^{w^{\cdot}_2}}{\left(1+m_2\right)^{w^{\cdot}_2}+\left(1-m_2\right)^{w^{\cdot}_2}}\right\}}{1-\left\{\frac{\left(1+m_1\right)^{w^{\cdot}_1}-\left(1-m_1\right)^{w^{\cdot}_1}}{\left(1+m_1\right)^{w^{\cdot}_1}+\left(1-m_1\right)^{w^{\cdot}_1}}\right\}.\left\{\frac{\left(1+m_2\right)^{w^{\cdot}_2}-\left(1-m_2\right)^{w^{\cdot}_2}}{\left(1+m_2\right)^{w^{\cdot}_2}+\left(1-m_2\right)^{w^{\cdot}_2}}\right\}}\right\},\\ \left\{\frac{-\left(\frac{-2\left(-m_1\right)^{w^{\cdot}_1}}{\left(-\left(-2-m_1\right)\right)^{w^{\cdot}_1}-\left(-\left(-m_1\right)^{w^{\cdot}_1}\right)}\right).\left(\frac{-2\left(-m_2\right)^{w^{\cdot}_2}}{\left(-\left(-2-m_2\right)\right)^{w^{\cdot}_2}-\left(-\left(-m_2\right)^{w^{\cdot}_2}\right)}\right)}{-1-\left(-1-\left(\frac{-2\left(-m_1\right)^{w^{\cdot}_1}}{\left(-\left(-2-m_1\right)\right)^{w^{\cdot}_1}-\left(-\left(-m_1\right)^{w^{\cdot}_1}\right)}\right)\right).\left(-1-\left(\frac{-2\left(-m_2\right)^{w^{\cdot}_2}}{\left(-\left(-2-m_2\right)\right)^{w^{\cdot}_2}-\left(-\left(-m_2\right)^{w^{\cdot}_2}\right)}\right)\right)}\right\}\end{array}\right\}$$

$$=\left\{\left\{\frac{\prod_{j=1}^{2}(1+m_j)^{w_j}-\prod_{j=1}^{2}(1-m_j)^{w_j}}{\prod_{j=1}^{2}(1+m_j)^{w_j}-\prod_{j=1}^{2}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{2}(-m_j)^{w_j}}{\prod_{j=1}^{2}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{2}(-m_j)^{w_j})}\right\}\right\}$$

Now we suppose that the given result holds for n=k

$$\oplus_{j=1}^{k}\left(w^{\cdot}_jH_j\right)=\left\{\left\{\frac{\prod_{j=1}^{k}(1+m_j)^{w_j}-\prod_{j=1}^{k}(1-m_j)^{w_j}}{\prod_{j=1}^{k}(1+m_j)^{w_j}-\prod_{j=1}^{k}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{k}(-m_j)^{w_j}}{\prod_{j=1}^{k}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{k}(-m_j)^{w_j})}\right\}\right\}$$

Now we prove that the result holds true for n=k+1

$$\oplus_{j=1}^{k+1}\left(w^{\cdot}_iH_i\right)=\oplus_{j=1}^{k}\left(w^{\cdot}_iH_i\right)\oplus w^{\cdot}_{k+1}H_{k+1}$$

$$=\left\{\begin{array}{l}\left\{\left\{\frac{\prod_{j=1}^{k}(1+m_j)^{w_j}-\prod_{j=1}^{k}(1-m_j)^{w_j}}{\prod_{j=1}^{k}(1+m_j)^{w_j}-\prod_{j=1}^{k}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{k}(-m_j)^{w_j}}{\prod_{j=1}^{k}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{k}(-m_j)^{w_j})}\right\}\right\}\oplus\\ \left\{\left\{\frac{(1+m_{k+1})^{w_{k+1}}-(1-m_{k+1})^{w_{k+1}}}{(1+m_{k+1})^{w_{k+1}}+(1-m_{k+1})^{w_{k+1}}}\right\},\left\{\frac{-2(-m_{k+1})^{w_{k+1}}}{(-(-2-m_{k+1}))^{w_{k+1}}-(-2(-m_{k+1})^{w_{k+1}})}\right\}\right\}\end{array}\right\}$$

$$=\left\{\left\{\frac{\prod_{j=1}^{k+1}(1+m_j)^{w_j}-\prod_{j=1}^{k+1}(1-m_j)^{w_j}}{\prod_{j=1}^{k+1}(1+m_j)^{w_j}-\prod_{j=1}^{k+1}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{k+1}(-m_j)^{w_j}}{\prod_{j=1}^{k+1}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{k+1}(-m_j)^{w_j})}\right\}\right\}$$

Hence the result holds true for n=k+1 so the result holds true for all n and we have

$$BVHFEWA\left(H_1,H_2,\ldots H_n\right)=\left\{\left\{\frac{\prod_{j=1}^{n}(1+m_j)^{w_j}-\prod_{j=1}^{n}(1-m_j)^{w_j}}{\prod_{j=1}^{n}(1+m_j)^{w_j}-\prod_{j=1}^{n}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{n}(-m_j)^{w_j}}{\prod_{j=1}^{n}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{n}(-m_j)^{w_j})}\right\}\right\}$$

Definition 9:

For some BVHFEs H_i and their weight vectors $w^{\cdot T}_i$ such that $w^{\cdot}_i\in[0,1]$ and $\sum_{j=1}^{n}w^{\cdot}_j=1$. The BVHFEWG operator is a function $\Psi^n\to\Psi$ such that

$$BVHFEWG\left(H_1,H_2,\ldots H_n\right)=\otimes_{j=1}^{n}\left(H_j\right)^{w^{\cdot}_j}$$

If we choose the weight vector, $w^{\cdot}=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\frac{1}{n},\ldots\frac{1}{n}\right)^T$ then BVHFEWG operator becomes BVHFEG operator i.e.

$$BVHFEG\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)=\underset{j=1}{\overset{n}{\otimes}}\left(\mathrm{H}_j\right)^{\frac{1}{n}}$$

Theorem 2:

For some BVHFEs H_i. Their aggregate calculated by using BVHFEWG operator is a BVHFE and

$$BVHFEWG\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)=$$

$$=\left\{\frac{2\prod_{j=1}^{n}(m_j)^{w_j}}{\prod_{j=1}^{n}(2-m_j)^{w_j}-\prod_{j=1}^{n}(m_j)^{w_j}}\right\},\left\{\frac{-\left(\prod_{j=1}^{n}\left(-\left(-1-\left(m_j\right)\right)\right)^{w_j}-\prod_{j=1}^{n}\left(-\left(-1-\left(m_j\right)\right)\right)^{w_j}\right)}{\prod_{j=1}^{n}\left(-\left(-2-\left(m_j\right)\right)\right)^{w_j}-\left(-\left(\prod_{j=1}^{n}\left(-\left(m_j\right)\right)^{w_j}\right)\right)}\right\}$$

This result can be proved in analogous way by principle of mathematical induction as done in previous result.

Example 2:

Let $\mathrm{H}_1=\{\{0.5,0.6\},\{-0.2,-0.1\}\}$ and $\mathrm{H}_2=\{\{0.9,0.8\},\{-0.5,-0.4\}\}$ be two BVHFEs and let $\dot{w}=(0.35,0.65)^{\frac{1}{T}}$ then we calculate their aggregated value by BVHFEWA operator and BVHFWG operator as follows:

$$BVHFEWA(H_1,H_2)=\left\{\left\{\frac{\prod_{j=1}^{2}(1+m_j)^{w_j}-\prod_{j=1}^{2}(1-m_j)^{w_j}}{\prod_{j=1}^{2}(1+m_j)^{w_j}-\prod_{j=1}^{2}(1-m_j)^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{2}(-m_j)^{w_j}}{\prod_{j=1}^{2}(-(-2-m_j))^{w_j}-(-2\prod_{j=1}^{2}(-m_j)^{w_j})}\right\}\right\}$$

$$=\{\{0.817489,0.719378,0.833516,0.742801\},\{-0.09103,-0.06262,-0.06475,-0.04359\}\}$$

$$BVHFEWG\left(\mathrm{H}_1,\mathrm{H}_2\right)=\left\{\left\{\frac{2\prod_{j=1}^{2}(m_j)^{w_j}}{\prod_{j=1}^{2}(2-m_j)^{w_j}-\prod_{j=1}^{2}(m_j)^{w_j}}\right\},\left\{\frac{-\left(\prod_{j=1}^{2}\left(-\left(-1-\left(m_j\right)\right)\right)^{w_j}-\prod_{j=1}^{2}\left(-\left(-1-\left(m_j\right)\right)\right)^{w_j}\right)}{\prod_{j=1}^{2}\left(-\left(-2-\left(m_j\right)\right)\right)^{w_j}-\left(-\left(\prod_{j=1}^{2}\left(-\left(m_j\right)\right)^{w_j}\right)\right)}\right\}\right\}$$

$$=\{\{0.748068,0.686851,0.789691,0.727046\},\{-0.64554,-0.48977,-0.54397,-0.40964\}\}$$

Definition 10:

For some BVHFEs H_i,($H_{\sigma(j)}$ is the i[th] largest among them) and their weight vectors $w_i^{\cdot T}$ such that $w^{\cdot}_i\in[0,1]$ and $\sum_{j=1}^{n}w^{\cdot}_i=1$.

1. A BVHFEOWA operator is a function $BVHFEOWA:\Psi^n\to\Psi$, such that

$$BVHFEOWA\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)=\underset{j=1}{\overset{n}{\oplus}}\left(w^{\cdot}_j H_{\sigma(j)}\right)=w^{\cdot}_1 H_{\sigma(1)}+w^{\cdot}_2 H_{\sigma(2)}+\ldots w^{\cdot}_n H_{\sigma(n)}$$

2. A BVHFEOWG operator is a function $BVHFEOWG:\Psi^n\to\Psi$, such that

$$BVHFEOWG\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)=\underset{j=1}{\overset{n}{\otimes}}\left(\mathrm{H}_{\sigma(j)}\right)^{w^{\cdot}_j}=\mathrm{H}_{\sigma(1)}^{w^{\cdot}_1}\times\mathrm{H}_{\sigma(2)}^{w^{\cdot}_2}\times\mathrm{H}_{\sigma(3)}^{w^{\cdot}_3}\times\ldots\mathrm{H}_{\sigma(n)}^{w^{\cdot}_n}$$

Theorem 3:

For some BVHFEs H_i, their aggregate calculated by using BVHFEOWA operator or BVHFEOWG operator is a BVHFE and

1. $BVHFEOWA\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)$

$$=\bigcup_{m_{\sigma(j)}\in\mathrm{H}_{\sigma(j)}}\left\{\left\{\frac{\prod_{j=1}^{n}(1+m_{\sigma(j)})^{w_j}-\prod_{j=1}^{n}(1-m_{\sigma(j)})^{w_j}}{\prod_{j=1}^{n}(1+m_{\sigma(j)})^{w_j}-\prod_{j=1}^{n}(1-m_{\sigma(j)})^{w_j}}\right\},\left\{\frac{-2\prod_{j=1}^{n}(-m_{\sigma(j)})^{w_j}}{\prod_{j=1}^{n}(-(-2-m_{\sigma(j)}))^{w_j}-(-2\prod_{j=1}^{n}(-m_{\sigma(j)})^{w_j})}\right\}\right\}$$

2. $BVHFEOWG\left(\mathrm{H}_1,\mathrm{H}_2,\ldots\mathrm{H}_n\right)$

$$=\bigcup_{m_{\sigma(j)}\in\mathrm{H}_{\sigma(j)}}\left\{\left\{\frac{2\prod_{j=1}^{n}(m_{\sigma(j)})^{w_j}}{\prod_{j=1}^{n}(2-m_{\sigma(j)})^{w_j}-\prod_{j=1}^{n}(m_{\sigma(j)})^{w_j}}\right\},\left\{\frac{-\left(\prod_{j=1}^{n}\left(-\left(-1-\left(m_{\sigma(j)}\right)\right)\right)^{w_j}-\prod_{j=1}^{n}\left(-\left(-1-\left(m_{\sigma(j)}\right)\right)\right)^{w_j}\right)}{\prod_{j=1}^{n}\left(-\left(-2-\left(m_{\sigma(j)}\right)\right)\right)^{w_j}-\left(-\left(\prod_{j=1}^{n}\left(-\left(m_{\sigma(j)}\right)\right)^{w_j}\right)\right)}\right\}\right\}$$

Example 3:

Let $\mathrm{H}_1=\{\{0.1,0.2\},\{-0.3,-0.2\}\}$, $\mathrm{H}_2=\{\{0.5,0.6\},\{-0.2,-0.1\}\}$ and $\mathrm{H}_3=\{\{0.9,0.8\},\{-0.2,-0.1\}\}$ be three BVHFEs and let $\dot{w}=(0.3,0.5,0.2)^{\frac{1}{T}}$ be the aggregation- associated weight vector. Then

$$\mathcal{S}\left(\mathrm{H}_1\right)=\frac{1}{\ell_{\mathrm{H}_1}}\left(\xi^+_{\mathrm{H}_1}+\xi^-_{\mathrm{H}_1}\right)=\frac{1}{2}\left(0.1+0.2+(-0.3)+(-0.2)\right)=-0.1$$

$$\mathcal{S}\left(\mathrm{H}_2\right)=\frac{1}{\ell_{\mathrm{H}_2}}\left(\xi^+_{\mathrm{H}_2}+\xi^-_{\mathrm{H}_2}\right)=\frac{1}{2}\left(0.5+0.6+(-0.2)+(-0.1)\right)=0.8$$

$$\mathcal{S}\left(\mathrm{H}_3\right)=\frac{1}{\ell_{\mathrm{H}_3}}\left(\xi^+_{\mathrm{H}_3}+\xi^-_{\mathrm{H}_3}\right)=\frac{1}{2}\left(0.9+0.8+(-0.2)+(-0.1)\right)=0.7$$

Clearly as

$$\mathcal{S}\left(\mathrm{H}_1\right)<\mathcal{S}\left(\mathrm{H}_3\right)<\mathcal{S}\left(\mathrm{H}_2\right)$$

So

$H_{\sigma(1)}=\mathrm{H}_2=\{\{0.5,0.6\},\{-0.2,-0.1\}\}$, $H_{\sigma(2)}=\mathrm{H}_3=\{\{0.9,0.8\},\{-0.2,-0.1\}\}$ *and* $H_{\sigma(3)}=\mathrm{H}_1=\{\{0.1,0.2\},\{-0.3,-0.2\}\}$

Now

$$BVHFEOWA\left(\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3\right)=\left\{\left\{\begin{matrix}0.7264,0.7359,0.6256,0.6379,\\0.7461,0.7551,0.6512,0.6628\end{matrix}\right\},\left\{\begin{matrix}-0.2173,-0.2,-0.1548,-0.1421,\\-0.1775,-0.1631,-0.1257,-0.1152\end{matrix}\right\}\right\}$$

$$BVHFEOWG\left(\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3\right)=\left\{\left\{\begin{matrix}0.5305,0.5908,0.4916,0.5491,\\0.5604,0.6226,0.5201,0.5796\end{matrix}\right\},\left\{\begin{matrix}-0.2819,-0.2500,-0.2038,-0.1759,\\-0.2334,-0.2041,-0.1624,-0.1361\end{matrix}\right\}\right\}$$

Here we see that BVHFEWA operator and BVHFEWG operators weight the BVHF argument but it gives no importance to the order of argument while BVHFEOWA operator and BVHFOWG operator give weight to the ordered position of the statement and ignore its worth. Enhancing these concepts, we are going to introduce the concept of hybrid operators that weight the ordered position of statement along with its importance.

Definition 11:

For some BVHFEs $\mathrm{H_i}$ and their weight vectors $w^{\cdot T}_{i}$ such that $w_i \in [0,1]$ *and* $\sum_{j=1}^{n} w_i = 1$. Here '**n**' is the balancing coefficient and $w^{\cdot T}_{i}$ be the weight vector for aggregation process of $\mathrm{H_i}$ with $w_i \in [0,1]$ *and* $\sum_{j=1}^{n} w_i = 1$.

1. The BVHFEHA operator is a mapping $BVHFEOWA : \Psi^n \to \Psi$ such that

$$\mathrm{BPVHFEHA}\left(\dot{\mathrm{H}}_1, \dot{\mathrm{H}}_2, \ldots \dot{\mathrm{H}}_n\right) = \overset{n}{\underset{j=1}{\oplus}} \left(w_j \dot{H}_{\sigma(j)} \right) = w_1 \dot{H}_{\sigma(1)} + w_2 \dot{H}_{\sigma(2)} + \ldots w_n \dot{H}_{\sigma(n)}$$

where $\dot{H}_{\sigma(j)}$ is the r[th] largest of $\dot{H} = nw_k H_k \left(k = 1,2,3,\ldots,n\right)$

2. e BVHFEHG operator is a mapping $BVHFEOWA : \Psi^n \to \Psi$ such that

$$\mathrm{BVHFEHG}\left(\ddot{\mathrm{H}}_1, \ddot{\mathrm{H}}_2, \ldots \ddot{\mathrm{H}}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\ddot{\mathrm{H}}_{\sigma(j)} \right)^{w^{\cdot}_{j}} = \ddot{\mathrm{H}}_{\sigma(1)} \times \ddot{\mathrm{H}}_{\sigma(2)}^{\;w^{\cdot}_{2}} \times \ldots \times \ddot{\mathrm{H}}_{\sigma(n)}^{\;w^{\cdot}_{n}}$$

where $\dot{H}_{\sigma(j)}$ is the r[th] largest of $\ddot{\mathrm{H}} = H_k^{nw_k} \left(k = 1,2,3,\ldots,n\right)$

Theorem 4:

For some BVHFEs $\mathrm{H_i}$, their aggregate calculated by using BVHFEHA operator or BVHFEHG operator is a BVHFE and

1. $\mathrm{BPVHFEHA}\left(\dot{\mathrm{H}}_1, \dot{\mathrm{H}}_2, \ldots \dot{\mathrm{H}}_n\right)$

$$= \bigcup_{\dot{m}_{\sigma(i)} \in \dot{\mathrm{H}}_{\sigma(i)}} \left\{ \left\{ \frac{\prod_{j=1}^{n} (1 + \dot{m}_{\sigma(i)})^{w_j} - \prod_{j=1}^{n} (1 - \dot{m}_{\sigma(i)})^{w_j}}{\prod_{j=1}^{n} (1 + \dot{m}_{\sigma(i)})^{w_j} - \prod_{j=1}^{n} (1 - \dot{m}_{\sigma(i)})^{w_j}} \right\}, \left\{ \frac{-2\prod_{j=1}^{n} (-\dot{m}_{\sigma(i)})^{w_j}}{\prod_{j=1}^{n} (-(-2 - \dot{m}_{\sigma(i)}))^{w_j} - (-2\prod_{j=1}^{n} (-\dot{m}_{\sigma(i)})^{w_j})} \right\} \right\}$$

2. $\mathrm{BVHFEHG}\left(\ddot{\mathrm{H}}_1, \ddot{\mathrm{H}}_2, \ldots \ddot{\mathrm{H}}_n\right)$

$$= \bigcup_{\ddot{m}_{\sigma(i)} \in \ddot{\mathrm{H}}_{\sigma(i)}} \left\{ \left\{ \frac{2\prod_{j=1}^{n} (\ddot{m}_{\sigma(i)})^{w_j}}{\prod_{j=1}^{n} (2 - \ddot{m}_{\sigma(i)})^{w_j} - \prod_{j=1}^{n} (\ddot{m}_{\sigma(i)})^{w_j}} \right\}, \left\{ \frac{-\left(\prod_{j=1}^{n} \left(-\left(-1 - \left(\ddot{m}_{\sigma(i)} \right) \right) \right)^{w_j} - \prod_{j=1}^{n} \left(-\left(-1 - \left(\ddot{m}_{\sigma(i)} \right) \right) \right)^{w_j} \right)}{\prod_{j=1}^{n} \left(-\left(-2 - \left(\ddot{m}_{\sigma(i)} \right) \right) \right)^{w_j} - \left(-\left(\prod_{j=1}^{n} \left(-\left(\ddot{m}_{\sigma(i)} \right) \right)^{w_j} \right) \right)} \right\} \right\}$$

Example 4:

Let $\mathrm{H}_1 = \left\{\{0.1, 0.2\}, \{-0.3, -0.2\}\right\}$, $\mathrm{H}_2 = \left\{\{0.5, 0.6\}, \{-0.2, -0.1\}\right\}$ and $\mathrm{H}_3 = \left\{\{0.9, 0.8\}, \{-0.2, -0.1\}\right\}$ be three BVHFEs and $w = \left(0.15, 0.2, 0.65\right)$ be their weight vector and let $\dot{w} = \left(0.3, 0.5, 0.2\right)^{\frac{1}{T}}$ be the aggregation- associated vector. Then

$$\dot{\mathrm{H}}_1 = \left\{\left\{1 - \left(1 - 0.1\right)^{2\times0.15}, 1 - \left(1 - 0.2\right)^{2\times0.15}\right\}, \left\{-\left(-\left(-0.3\right)\right)^{2\times0.15}, -\left(-\left(-0.2\right)\right)^{2\times0.15}\right\}\right\}$$

$$\dot{\mathrm{H}}_1 = \left\{\{0.031114, 0.064752\}, \{-0.69685, -0.61703\}\right\}$$

$$\dot{\mathrm{H}}_2 = \left\{\left\{1 - \left(1 - 0.5\right)^{2\times0.2}, 1 - \left(1 - 0.6\right)^{2\times0.2}\right\}, \left\{-\left(-\left(-0.2\right)\right)^{2\times0.2}, -\left(-\left(-0.1\right)\right)^{2\times0.2}\right\}\right\}$$

$$\dot{\mathrm{H}}_2 = \left\{\{0.242142, 0.306855\}, \{-0.52531, -0.39811\}\right\}$$

$$\dot{\mathrm{H}}_3 = \left\{\left\{1 - \left(1 - 0.9\right)^{2\times0.65}, 1 - \left(1 - 0.8\right)^{2\times0.65}\right\}, \left\{-\left(-\left(-0.2\right)\right)^{2\times0.65}, -\left(-\left(-0.1\right)\right)^{2\times0.65}\right\}\right\}$$

$$\dot{\mathrm{H}}_3 = \left\{\{0.949881, 0.876593\}, \{-0.12341, -0.05012\}\right\}$$

The score values are:

$$\mathcal{S}\left(\dot{\mathrm{H}}_1\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_1}}\left(\xi^{+}_{\dot{\mathrm{H}}_1} + \xi^{-}_{\dot{\mathrm{H}}_1}\right) = \frac{1}{2}\left(0.031114 + 0.064752 + \left(-0.69685\right) + \left(-0.61703\right)\right)$$

$$\mathcal{S}\left(\dot{\mathrm{H}}_1\right) = -0.60901$$

$$\mathcal{S}\left(\dot{\mathrm{H}}_2\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_2}}\left(\xi^{+}_{\dot{\mathrm{H}}_2} + \xi^{-}_{\dot{\mathrm{H}}_2}\right) = \frac{1}{2}\left(0.242142 + 0.306855 + \left(-0.52531\right) + \left(-0.39811\right)\right)$$

$$\mathcal{S}\left(\dot{\mathrm{H}}_2\right) = -0.18721$$

$$\mathcal{S}\left(\dot{\mathrm{H}}_3\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_3}}\left(\xi^{+}_{\dot{\mathrm{H}}_3} + \xi^{-}_{\dot{\mathrm{H}}_3}\right) = \frac{1}{2}\left(0.949881 + 0.876593 + \left(-0.12341\right) + \left(-0.05012\right)\right)$$

$$\mathcal{S}\left(\dot{\mathrm{H}}_3\right) = 0.826472$$

Now from these results it is obvious that

$\mathcal{S}\left(\dot{\mathrm{H}}_3\right) > \mathcal{S}\left(\dot{\mathrm{H}}_2\right) > \mathcal{S}\left(\dot{\mathrm{H}}_1\right)$, so

$$\dot{H}_{\sigma(1)} = \dot{\mathrm{H}}_3 = \left\{\{0.949881, 0.876593\}, \{-0.12341, -0.05012\}\right\}$$

$$\dot{H}_{\sigma(2)} = \dot{\mathrm{H}}_2 = \left\{\{0.242142, 0.306855\}, \{-0.52531, -0.39811\}\right\}$$

$$\dot{H}_{\sigma(3)} = \dot{\mathrm{H}}_1 = \left\{\{0.031114, 0.064752\}, \{-0.69685, -0.61703\}\right\}$$

Now

$$BVHFEHA\left(\dot{\mathrm{H}}_1, \dot{\mathrm{H}}_2, \dot{\mathrm{H}}_3\right) = \left\{ \begin{array}{l} \{0.5908, 0.5952, 0.6131, 0.6173, 0.4915, 0.4966, 0.5176, 0.5225\}, \\ \{-0.3776, -0.3666, -0.3255, -0.3158, -0.2987, -0.2896, -0.2558, -0.2478\} \end{array} \right\}$$

Now we calculate $\ddot{H}_1, \ddot{H}_2, \ddot{H}_2$

$$\ddot{H}_1 = \left\{\left\{(0.1)^{2\times0.15}, (0.2)^{2\times0.15}\right\}, \left\{-1 - \left(-\left(-\left(-1 - (-0.3)\right)\right)^{2\times0.15}\right), -1 - \left(-\left(-\left(-1 - (-0.2)\right)\right)^{2\times0.15}\right)\right\}\right\}$$

$$\ddot{H}_1 = \left\{\{0.501187, 0.617034\}, \{-0.10148, -0.06475\}\right\}$$

$$\ddot{H}_2 = \left\{\left\{(0.5)^{2\times0.2}, (0.6)^{2\times0.2}\right\}, \left\{-1 - \left(-\left(-\left(-1 - (-0.2)\right)\right)^{2\times0.2}\right), -1 - \left(-\left(-\left(-1 - (-0.1)\right)\right)^{2\times0.2}\right)\right\}\right\}$$

$$\ddot{H}_2 = \left\{\{0.757858, 0.815193\}, \{-0.08539, -0.04127\}\right\}$$

$$\ddot{H}_3 = \left\{\left\{(0.9)^{2\times0.65}, (0.8)^{2\times0.65}\right\}, \left\{-1 - \left(-\left(-\left(-1 - (-0.2)\right)\right)^{2\times0.65}\right), -1 - \left(-\left(-\left(-1 - (-0.1)\right)\right)^{2\times0.65}\right)\right\}\right\}$$

$$\ddot{H}_3 = \left\{\{0.871998, 0.748199\}, \{-0.2518, -0.128\}\right\}$$

The score values are:

$$\mathcal{S}\left(\ddot{H}_1\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_1}}\left(\xi^{+}_{\dot{\mathrm{H}}_1} + \xi^{-}_{\dot{\mathrm{H}}_1}\right) = \frac{1}{2}\left(0.501187 + 0.617034 + \left(-0.10148\right) + \left(-0.06475\right)\right)$$

$$\mathcal{S}\left(\ddot{H}_1\right) = 0.475996$$

$$\mathcal{S}\left(\ddot{H}_2\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_2}}\left(\xi^{+}_{\dot{\mathrm{H}}_2} + \xi^{-}_{\dot{\mathrm{H}}_2}\right) = \frac{1}{2}\left(0.757858 + 0.815193 + \left(-0.08539\right) + \left(-0.04127\right)\right)$$

$$\mathcal{S}\left(\ddot{H}_2\right) = 0.723196$$

$$\mathcal{S}\left(\ddot{H}_3\right) = \frac{1}{\ell_{\dot{\mathrm{H}}_3}}\left(\xi^{+}_{\dot{\mathrm{H}}_3} + \xi^{-}_{\dot{\mathrm{H}}_3}\right) = \frac{1}{2}\left(0.871998 + .748199 + \left(-0.2518\right) + \left(-0.128\right)\right)$$

$$\mathcal{S}\left(\ddot{H}_3\right) = 0.620199$$

Now from these results it is obvious that

$S(\ddot{H}_2) > S(\ddot{H}_3) > S(\ddot{H}_1)$, so

$$\ddot{H}_{\sigma(1)} = \ddot{H}_2 = \{\{0.757858, 0.815193\}, \{-0.08539, -0.04127\}\}$$

$$\ddot{H}_{\sigma(2)} = \ddot{H}_3 = \{\{0.871998, 0.748199\}, \{-0.2518, -0.128\}\}$$

$$\ddot{H}_{\sigma(3)} = \ddot{H}_1 = \{\{0.501187, 0.617034\}, \{-0.10148, -0.06475\}\}$$

$$BVHFEHG(\ddot{H}_1, \ddot{H}_2, \ddot{H}_3) = \left\{ \begin{array}{l} \{0.7569, 0.7843, 0.6974, 0.7239, 0.7740, 0.8015, 0.7139, 0.7406\}, \\ \{-0.2059, -0.1949, -0.1234, -0.1141, -0.1855, -0.1753, -0.1065, -0.0977\} \end{array} \right\}$$

Application

In this section, we applied Einstein operation for BVHFS on a DM problem. First, we construct a DM matrix in which decision makers characterized every element by a BVHFE. The brief decision making process is as follows:

Consider that we have n substitutes A_i with m attributes x_j where $(i, j = 1, 2, 3, \ldots m)$ and and assume that $w = (w_1, w_2, \ldots, w_m)$ be the weight vector such that $w_j \in [-1,1]$, $j = 1, 2, 3 \ldots m$ and $\sum_{j=1}^{m} w_j = 1$. The analysts take values for the substitutes A_i under the attributes x_j in the form of BVHFEs (H_{ij}) anonymously.

This method is illustrated as follows:

Step 1:

In this step decision matrix is established.

Step 2:

In this step, BVHFE $\alpha_i (i = 1, 2, 3, \ldots, n)$ can be obtained for the substitutes $A_i (i = 1, 2, 3, \ldots, n)$ by defined aggregation operators.

Step 3:

In this step, score values are calculated.

Step 4:

Based on score analysis, we get the best alternatives.

Example 5:

A high school needs a mathematics lecturer for their secondary classes. The school management advertised the post for a lecturer in the newspaper and several master degree holders applied for it. The management of school will have to appoint only one candidate that is most suitable. The board for the appointment of lecturer consists of the CEO of school, the principal of school, and the director academics of school. The school is looking for a well-experienced, hard worker and a skillful subject specialist for the post described above. After the posting of advertisement in the newspaper, a number of qualified persons applied for the post. Initial screening tests were organized and three candidates were selected for interview.

Let $A = \{A_1, A_2, A_3\}$ be the set of substitutes and $X = \{x_1, x_2, x_3\}$ be the set of attributes and let $w = (0.25, 0.35, 0.40)^T$ be the weight vector of the attributes $X_i (i = 1, 2, 3)$ and be $w' = (0.3, 0.5, 0.2)^{\frac{1}{T}}$ the aggregation associated weight vector.

Now for the selection of most suitable candidate, we use DM method.

Step 1:

Avoiding any kind of influence, the decision makers, in the state of being anonymous, construct the decision matrix shown in the table below.

Table 1 (Decision Matrix)

	x1	x2	x3
A_1	$H_{11} = \{\{0.4, 0.5\}, \{-0.3, -0.2\}\}$	$H_{12} = \{\{0.6, 0.7\}, \{-0.1, -0.2\}\}$	$H_{13} = \{\{0.3, 0.4\}, \{-0.6, -0.7\}\}$
A_2	$H_{21} = \{\{0.6, 0.8\}, \{-0.4, -0.6\}\}$	$H_{22} = \{\{0.4, 0.6\}, \{-0.3, -0.1\}\}$	$H_{23} = \{\{0.7, 0.9\}, \{-0.4, -0.2\}\}$
A_3	$H_{31} = \{\{0.3, 0.5\}, \{-0.6, -0.5\}\}$	$H_{32} = \{\{0.5, 0.8\}, \{-0.5, -0.3\}\}$	$H_{33} = \{\{0.4, 0.6\}, \{-0.1, -0.2\}\}$

Step 2 (a):

We now aggregate the values of given decision matrix and calculate respectively by using BVHFEWA operator.

$$H_1 = BVHFEWA(H_{11}, H_{12}, H_{13})$$

$$H_1 = \left\{ \begin{array}{l} \{0.4896, 0.5530, 0.5686, 0.5178, 0.5343, 0.5786, 0.5936\}, \\ \{-0.2064, -0.2150, -0.2865, -0.2979, -0.1821, -1898, -0.2541, -0.2644\} \end{array} \right\}$$

$$H_2 = \left\{ \begin{array}{l} \{0.5322, 0.6133, 0.6218, 0.6905, 0.6137, 0.6836, 0.6909, 0.7490\}, \\ \{-0.3472, -0.3031, -0.2058, -0.1777, -0.3960, -0.3470, -0.2376, -0.2057\} \end{array} \right\}$$

$$H_3 = \left\{ \begin{array}{l} \{0.4237, 0.4669, 0.6212, 0.6532, 0.4809, 0.5213, 0.6634, 0.6925\}, \\ \{-0.3981, -0.4478, -0.3062, -0.3470, -0.3746, -0.4222, -0.2872, -0.3259\} \end{array} \right\}$$

Step 3 (a):

We calculate the score value for H_1,H_2,H_3 respectively.

$$\mathcal{S}(H_1)=0.3057 \qquad \mathcal{S}(H_2)=0.3719 \qquad \mathcal{S}(H_3)=0.2018$$

Step 4 (a):

By ordering the score values we obtained the result as follows:

As $\mathcal{S}(H_2)>\mathcal{S}(H_1)>\mathcal{S}(H_3)$

So A_2 is the best candidate.

We now aggregate the values of given decision matrix and calculate H_1,H_2,H_3 respectively by using BPVHFEWG operator.

Step 2 (b):

$$H_1 = BVHFEWG(H_{11},H_{12},H_{13})$$

$$H_1=\begin{Bmatrix}\{0.4678, 0.4932, 0.5099, 0.5368, 0.4994, 0.5260, 0.5434, 0.5714\}, \\ \{-0.3509, -0.3950, -0.4516, -0.5017, -0.3022, -0.3445, -0.3929, -0.4400\}\end{Bmatrix}$$

$$H_2=\begin{Bmatrix}\{0.5104, 0.5428, 0.6193, 0.6557, 0.5624, 0.5968, 0.6774, 0.7154\}, \\ \{-0.5381, -0.4482, -0.3279, -0.2638, -0.6959, -0.5881, -0.4423, -0.3710\}\end{Bmatrix}$$

$$H_3=\begin{Bmatrix}\{0.4127, 0.4492, 0.5377, 0.5812, 0.4787, 0.5191, 0.6160, 0.6628\}, \\ \{-0.7873, -0.8752, -0.5328, -0.5881, -0.7002, -0.7769, -0.4643, -0.5140\}\end{Bmatrix}$$

Step 3 (b):

We calculate the score value for H_1,H_2,H_3 respectively.

$$\acute{\mathcal{S}}(H_1)=0.1211 \qquad \acute{\mathcal{S}}(H_2)=0.1506 \qquad \acute{\mathcal{S}}(H_3)=-0.1227$$

Step 4 (b):

By ordering the score values we obtained the result as follows:

As $\acute{\mathcal{S}}(H_2)>\acute{\mathcal{S}}(H_1)>\acute{\mathcal{S}}(H_3)$

So A_2 is the best candidate.

So either by using BPVHFEWA operator or by BPVHFEWG operator we observe that candidate is the most suitable among three.

CONCLUSION

In our treatise, we successfully define Einstein aggregation operators for BVHFSs and proved some useful results. We apply the newly defined operations on a DM problem and got the same result by using two different kinds of aggregation operators. Same results could be obtained if we use Einstein ordered weighted averaging or geometric operator or Einstein hybrid operator for BVHFSs.

REFERENCES

1. *Abdullah, S., Aslam, M., Ullah, K., (2014), "Bipolar fuzzy soft sets and its applications in decision making problem." Journal of Intelligent & Fuzzy Systems, 27(2): 729-742.*

2. *Atanassov, K., and Gargov, G., (1989), "Interval valued intuitionistic fuzzy sets." Fuzzy sets and Systems, 31(3): 343-349.*

3. *Atanassov, K. T., (1986), "Intuitionistic fuzzy sets." Fuzzy sets and Systems, 20(1): 87-96.*

4. *Chauhan, A. and Vaish, R., (2014), "A comparative study on decision making methods with interval data." Journal of Computational Engineering, 2014.*

5. *Chen, N. and Xu, Z., (2014), "Properties of interval-valued hesitant fuzzy sets." Journal of Intelligent & Fuzzy Systems, 27(1): 143-158.*

6. *Chen, N., Xu, Z., Xia, M., (2013), "Interval-valued hesitant preference relations and their applications to group decision making." Knowledge-Based Systems, 37: 528-540.*

7. *Chen, T.-Y., (2011), "A comparative analysis of score functions for multiple criteria decision making in intuitionistic fuzzy settings." Information Sciences, 181(17): 3652-3676.*

8. *De, S. K., Biswas, R., Roy., A.R., (2000). "Some operations on intuitionistic fuzzy sets." Fuzzy sets and Systems 114(3): 477-484.*

9. *Dubois, D. and Prade, H., (2001), Interval-valued Fuzzy Sets, Possibility Theory and Imprecise Probability.*

10. *Farhadinia, B., (2013), "Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets." Information Sciences 240: 129-144.*

11. *Herrera-Viedma, E., Alonso, S., Chiclana, F., Herrera,F., (2007), "A consensus model for group decision making with incomplete fuzzy preference relations." IEEE Transactions on fuzzy Systems, 15(5): 863-877*

12. *Hong, D. H., and Choi, C.-H., (2000), "Multicriteria fuzzy decision-making problems based on vague set theory." Fuzzy sets and Systems 114(1): 103-113.*

13. *Hwang, C.L., and Yoon, K., (1981), Multiple Attribute Decision Making: Methods and Applications. Springer-Verlag, New York.*

14. *Klutho, S., (2013), "Mathematical Decision Making" An Overview of the Analytic Hierarchy Process. Available online a t:www. whitman. edu/mathematics/ SeniorProjectArchive/2013/Klutho. Pdf.*

15. *Lee, K., (2000), "Bipolar-valued fuzzy sets and their operations", Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 2000.*

16. *Li, D.-F., (2005), "Multiattribute decision making models and methods using intuitionistic fuzzy sets." Journal of computer and System Sciences, 70(1): 73-85.*

17. *Liu, H.-W., and Wang, G.-J., (2007), "Multi-criteria decision-making methods based on intuitionistic fuzzy sets", European Journal of Operational Research, 179(1): 220-233.*

18. *Liu, P., Mahmood, T., Khan, T., (2017), "Multi-Attribute Decision-Making Based on Prioritized Aggregation Operator under Hesitant Intuitionistic Fuzzy Linguistic Environment" Symmetry, 9(11): 270.*

19. *Mahmood, T. and Munir, M., (2013), "On bipolar fuzzy subgroups" World Applied Sciences Journal, 27(12): 1806-1811.*

20. *Mahmood, T., Broumi, S., Ullah, K., Khan, Q., (2016). Bipolar-valued hesitant fuzzy sets and their application in multi-attribute decision making, submitted.*

21. *Ye, J., Mahmood, T., Khan, Q., (2016), "Vector similarity measures for simplified neutrosophic hesitant fuzzy set and their applications" Journal of Inequalities and Special Functions, 7(4): 176-194.*

22. *Mahmood, T., Smarandache, F., Ullah, K., Khan, Q., (2017). Some Aggregation Operators For Bipolar-Valued Hesitant Fuzzy Information.*

23. *Neves, R. D. S., and Livet, P., (2008), "Bipolarity in human reasoning and affective decision making", International Journal of Intelligent Systems, 23(8): 898-922.*

24. *Shabir, M., and Israr, A. K., (2008), "Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semigroups", Mathware & Soft Computing, 15(3): 263-272.*

25. *Torra, V., (2010), "Hesitant fuzzy sets", International Journal of Intelligent Systems, 25(6): 529-539.*

26. *Torra, V., and Narukawa, Y., (2009), On hesitant fuzzy sets and decision. Fuzzy Systems, FUZZ-IEEE 2009, IEEE International Conference on, IEEE.*

27. *Wang, W., and Liu, X., (2011), "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations" International Journal of Intelligent Systems, 26(11): 1049-1075.*

28. *Wang, W., and Liu, X., (2011), "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations", International Journal of Intelligent Systems, 26(11): 1049-1075.*

29. *Xia, M., and Xu, Z., (2011), "Hesitant fuzzy information aggregation in decision making", International journal of approximate reasoning, 52(3): 395-407.*

30. *Xia, M., Xu, Z., Chen, N., (2013), "Some hesitant*

fuzzy aggregation operators with their application in group decision making", Group Decision and Negotiation, 22(2): 259-279.

31. *Xu, Z., (2004), "A method based on linguistic aggregation operators for group decision making with linguistic preference relations", Information Sciences, 166(1): 19-30.*

32. *Xu, Z., (2007), "Intuitionistic fuzzy aggregation operators", IEEE Transactions on fuzzy Systems, 15(6): 1179-1187.*

33. *Xu, Z., and Yager, R. R., (2006), "Some geometric aggregation operators based on intuitionistic fuzzy sets", International journal of general systems, 35(4): 417-433.*

34. *Xu, Z.-S., and Jian, C., (2007), "Approach to group decision making based on interval-valued intuitionistic judgment matrices", Systems Engineering-Theory & Practice, 27(4): 126-133.*

35. *Ye, J., (2007), "Improved method of multicriteria fuzzy decision-making based on vague sets", Computer-Aided Design, 39(2): 164-169.*

36. *Ye, J., (2009), "Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment", Expert Systems with Applications, 36(3): 6899-6902.*

37. *Yu, D., (2014), "Some hesitant fuzzy information aggregation operators based on Einstein operational laws", International Journal of Intelligent Systems, 29(4): 320-340.*

38. *Zadeh, L. A., (1965), "Information and control" Fuzzy sets, 8(3): 338-353.*

39. *Zadeh, L. A., (1975), "The concept of a linguistic variable and its application to approximate reasoning—I", Information Sciences, 8(3): 199-249.*